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Numerical study of the behaviour of a system of parallel line vortices

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The dynamical behaviour of a system of parallel line vortices in an inviscid fluid is studied numerically. The initial configuration of the system is assumed to be such that the points of intersection of the line vortices with a plane normal to the vorticity form a regular polygon. The numerical experiments show that the vortex polygon is rearranged due to non-linear interactions among the line vortices in such a way as to produce a more or less uniform distribution of vortices inside the fluid with an approximately constant mean separation. The average angular velocity of the rotation of the vortex lines about the instantaneous centroid of the vortex system remains approximately constant. These results agree with the conjecture of Raja Gopal (1964). The results may prove to be of some value in a macroscopic model of liquid helium based on hydrodynamical principles.

1. Introduction

Liquid helium II at 0 °K has a peculiar behaviour (Andronikashvili & Mamaladze 1966; Pellam 1955, chapter 3). It has no viscosity but, at the same time, appears to take part in the rotation of the vessel in which it is placed. To explain these contradictory properties, Onsagar (1949) and Feynman (1955, chapter 2) have put forward the hypothesis that inside liquid helium, near absolute zero temperature, there appear quantized vortex lines when the containing vessel is set in rotation. The resultant motion of the fluid due to quantized vortex lines is such that the velocity field of the fluid about the axis of rotation corresponds to a 'solid-body rotation' of the vortices in the fluid (Reppy, Depatie & Lane 1960). Experimental observations on wave propagation in liquid helium have revealed unambiguously quantized velocity fields with the vorticity expected on the basis of the Onsagar–Feynman theory.

There are, however, some theoretical questions one should answer: (i) what is the actual mechanism by which these vortices are brought into existence; (ii) how do they interact amongst themselves and with the (boundary of the) container? In this note, we leave open the first question whose solution lies in a quantum mechanical description of the interaction between the liquid and the rotating boundary, but attempt to answer the second question using the Helmholtz theorem.

Several authors (Pincus & Shapiro 1965; Raja Gopal 1964; Fetter, Hohenberg & Pincus 1966; Tkachenko 1965, 1966) have studied, theoretically, the dynamics of parallel line vortex lattices in liquid helium II. However, there seems to be no agreement in their conclusions. Thus, for example, Fetter *et al.* conclude that any infinite lattice structure of the line vortices in liquid helium II is unstable; while Tkachenko concludes that an infinite array of line vortices forming a hexagonal lattice is stable and a square lattice is unstable. Raja Gopal proves that any random array of line vortices with a constant mean spacing is stable. There is yet no direct experimental evidence about the lattice structure of vortex lines in liquid helium II (Hall 1961, p. 580). The existing theoretical interpretations of the experimental data about collective effects of vortex lines deal with magnitudes averaged over a volume containing many vortex lines (Raja Gopal 1964), in which the essential quantity is the number of vortex lines per unit area normal to the direction of vorticity (Turkington, Brown & Osborne 1963). The common assumption made by the aforementioned authors in their theory is that the lattice of line vortices is infinite. This is not a realistic assumption because the velocity field due to such a system is indeterminate. This indeterminacy is removed if one considers a finite array of parallel line vortices.

Consider an assembly of N line vortices of unit strength parallel to Z axis, and let the point of intersection of the nth line vortex with a plane perpendicular to the Z axis be represented by a complex number, ξ_n, then according to the Helmholtz theorem, the complex conjugate of the velocity, $V(\xi_m)$, of the mth line vortex is given by

$$\overline{V(\xi_m)} = \sum_{n=0}^{N-1}{}' \frac{-i}{\xi_m - \xi_n}, \tag{1}$$

where the bar represents the complex conjugate, and the prime on summation indicates that $\xi_m \neq \xi_n$. It can then be shown (Milne-Thomson 1964) that the unbounded system of line vortices has a centroid which is stationary. There is no loss in generality in assuming that the origin of the complex plane coincides with the centroid. Now, the necessary and sufficient condition for the system of line vortices to rotate like a solid body with an angular velocity, Ω (Ω is real), is given by

$$V(\xi_m) = i\Omega\xi_m. \tag{2}$$

Using (1) and (2), we see that the condition for solid body rotation of the unbounded system of N line vortices is

$$\sum_{n=0}^{N-1}{}' \frac{1}{\xi_m - \xi_n} = \Omega\bar{\xi}_n \tag{3}$$

for all m. But (3) is not, in general, satisfied by an arbitrary system of parallel line vortices. If the ξ_n form a regular polygon, then (3) is satisfied and $\Omega = \frac{1}{2}(N-1)$. It can be shown that the polygon of line vortices in the fluid rotates like a solid body even when the fluid is enclosed in a concentric cylinder whose axes coincide with the centroid of the vortices. Havelock (1931) showed that when $N > 7$, the vortex system, whether bounded or unbounded by a coaxial cylinder, is unstable. He also studied the stability properties of the vortex system enclosed in a coaxial cylinder when $N \leqslant 7$.

Stauffer & Fetter (1968) computed the free energy of various equilibrium states of finite systems taking into account the effect of image vortices in a rotating cylinder filled with liquid helium II in order to determine the precise arrangement of vortices. They have shown that the vortices tend to form concentric circles about the centre of the cylinder. These authors also discuss the relation between minimum free energy and stability. However, their analysis does not reveal the evolution of the vortex system from a given initial state.

In this paper, we study the non-linear behaviour of the unstable vortex system when enclosed in a coaxial cylinder. The mathematical formulation is given in § 2. A discussion of the results of our numerical calculations is made in § 3, and the conclusions are summarized in § 4. All the calculations were performed on the digital computer, CDC 3600, of the National Computing Centre, Tata Institute of Fundamental Research, Bombay.

2. Non-linear theory

Let the co-ordinates of a system of N line vortices of strength K† in a fluid enclosed in a cylinder of radius, R, be given by $\xi_n = aZ_n$, where $a < R$ and, initially,

$$Z_n = \exp(2\pi in/N) \quad (n = 0, 1, 2, \ldots, (N-1)). \tag{4}$$

The complex conjugate velocity of the mth vortex is given by the following coupled system of non-linear differential equations (Milne-Thomson 1964)

$$\frac{d\bar{Z}_m}{d\tau} = \sum_{n=0}^{N-1}{}' \frac{-i}{(Z_m - Z_n)} + \sum_{n=0}^{N-1} \frac{i}{(Z_m - XZ_n/|Z_n|^2)} \quad \text{for} \quad m = 0, 1, 2, \ldots, (N-1), \tag{5}$$

where the prime on the first summation indicates that the terms $m = n$ are not included; Z_m is the non-dimensional complex co-ordinate of the mth line vortex, and X and τ are, respectively, defined as follows:

$$X = R^2/a^2, \tag{6}$$

and

$$\tau = tK/a^2. \tag{7}$$

The first summation on the right-hand side of (5) includes the interaction of vortices among themselves, and the second summation represents the effect of image vortices. When the system of equations (5) satisfies the Cauchy–Lipschitz condition, it has a unique solution for given initial values of Z_m (Struble 1962). If we linearize the equations (5) and study their stability properties, we find that the stability criteria are exactly the same as those obtained by Havelock

† We assume that the sign of vorticity is the same for all vortices, for the sake of simplicity of computation. The problem, when the vortices do not have the same sign, is interesting mathematically, but it is probably of no physical interest because the vorticity generated by the interaction of the fluid with the boundary is unlikely to lead to such a state.

(1931) for the bounded as well as unbounded systems. The non-linear system has been solved numerically by using the fourth-order Runge–Kutta method with Gill's modification for values of N equal to 2, 3, 4, 5, 10, 12, 18, 20, 30, 40 and 50 with pre-assigned initial values of Z_m which satisfy the Cauchy–Lipschitz condition. The values of X are so chosen that the system is unstable according to the linear theory of Havelock (1931). The solution is obtained with suitable time steps, $\Delta\tau$, such that the solution is unchanged when $\Delta\tau$ is reduced. After the solution to the system of equations (5) is found, the mean separation, M, of the vortices, using the formula

$$M = \frac{2 \sum\limits_{m,n=1}^{N}{}' |Z_m - Z_n|}{N(N-1)} \tag{8}$$

is calculated as a function of time. Some typical results, when $N = 2$, 5, 40 and 50, are shown in figures 1–4. In figure 5, we have given the distribution of the density of vortices (number of vortices per unit area *versus* radial distance) for the case $N = 50$ at time $\tau = 20$. In figures 6 and 7, we have shown the average, the minimum and the maximum of the modulus of the angular velocity of the vortex system around the instantaneous centre of rotation.

3. Discussion of numerical results

In all the calculations except the two cases mentioned below, the initial perturbation is such that all the vortices are displaced in the direction perpendicular to their axes, so that the real part of their co-ordinate is increased by 0·1. We refer to it hereafter as type A perturbation. (There is no specific reason, of course, to choose a perturbation value equal to 0·1. Any other value which fulfils the Cauchy–Lipschitz condition will do.) In a few cases, the initial perturbation is such that only one vortex is displaced, and the real part of its co-ordinate is increased by 0·05. We call this type B perturbation. It has been found that when $N \gtrsim 30$ the asymptotic behaviour of the solution is similar for these two types of perturbations. Figures 1–7 correspond to type A perturbation. In figure 1, which corresponds to $N = 2$, $X = 2$, we see that the mean separation of the vortex pair oscillates between the maximum and minimum values. Calculations were also done for $N = 3$ and 4, and the average separation (not shown in figures) was found to vary in some random manner with time. Figure 2 corresponds to $N = 5$, $X = 2$. When $N = 10$, the solution (not shown in figures) up to $\tau = 20$ (which is the upper limit to which the solution has been followed) shows the same qualitative behaviour for $X = 4$ or $X = \infty$. Presumably, it remains qualitatively the same even for large τ. We have computed the solution when $N = 18$ with $X = \infty$, and observed that the mean separation of the vortices increases with minor fluctuations, but definitely shows an average behaviour which appears to be a characteristic feature when N is of the order of 40 as shown in figure 3. Figures 3 and 4 correspond, respectively, to $N = 40$, $X = \infty$ and $N = 50$, $X = 4$. In these figures, one sees that the average separation of vortices increases with time without much fluctuation. One finds also that the rate of increase of the mean separation decreases with time, indicating the possibility that for larger times, the mean separation tends to a constant value. We see the same behaviour of the solution (not shown in figures) when $N = 20$, and also when $N = 30$. From these numerical experiments, it seems reasonable to conclude that, when N is sufficiently large, say, of the order of 40, the system of parallel line vortices, which is unstable according to linear theory, evolves in such a way that the mean separation increases but tends to a non-zero finite limit.

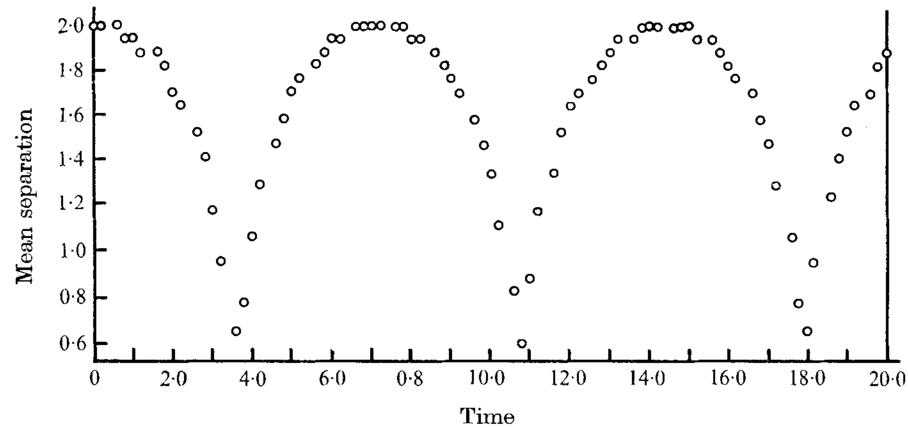

FIGURE 1. The mean separation of vortices is plotted against time, when $N = 2$, $X = 2$ and the time step of integration $\Delta\tau = 0{\cdot}05$. The initial perturbation is of type A.

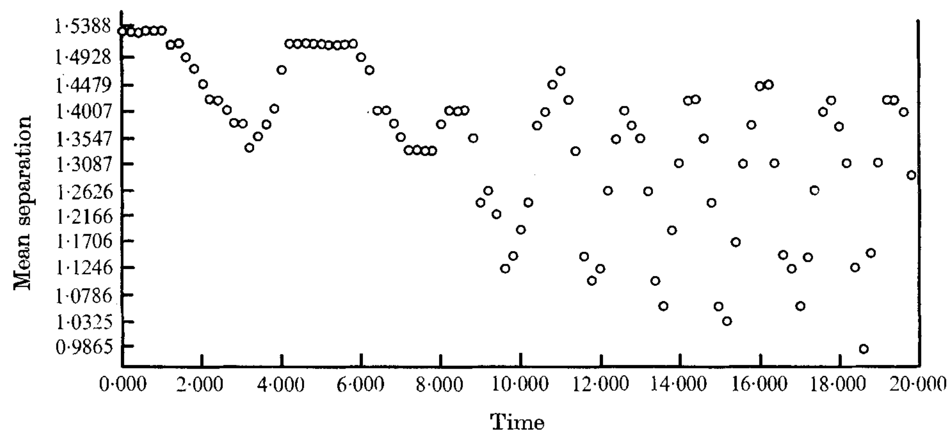

FIGURE 2. The mean separation of vortices is plotted against time, when $N = 5$, $X = 2$ and $\Delta\tau = 0{\cdot}002$. The initial perturbation is of type A.

In figure 5, we have shown the distribution of the density of vortices at $\tau = 20$ for the case $N = 50$, with $X = \infty$. This is calculated for the sake of definiteness by counting the number of vortices in an annulus of thickness 0·2. We have also obtained the distribution of the density of vortices at different times for both the cases when $N = 40$ and 50. From our calculations we found that the vortex distribution, which is initially zero inside the polygon for both N, rises to a value

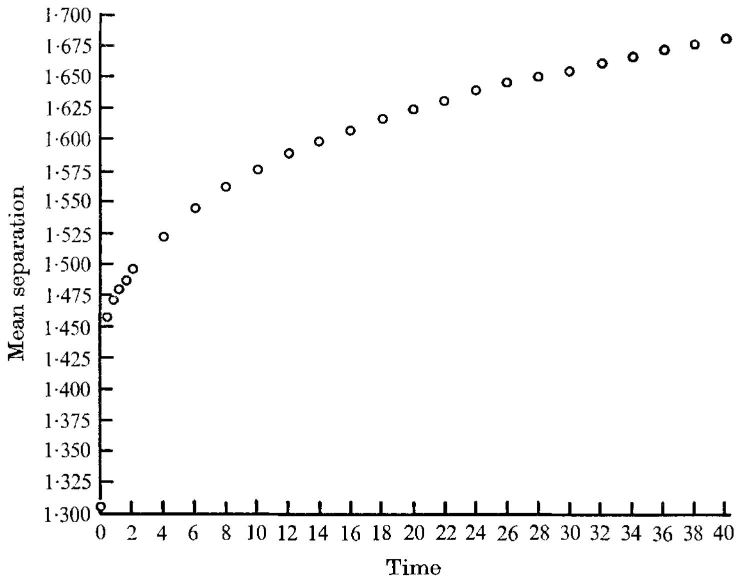

FIGURE 3. The mean separation of the vortices is plotted against time, when $N = 40$, $X = \infty$ and $\Delta\tau = 0{\cdot}04$. The initial perturbation is of type A.

comparable to the average value in a time of order of unity (measured in units of a^2/K). The same feature is observed when the fluid is bounded or unbounded.

The variation of the mean separation, with time for large N, is interesting. There are no large fluctuations in the mean separation. From these calculations, one observes that, when $\tau \gg 1$, a constant density of line vortices without much fluctuation will occur due to interactions amongst line vortices.

In figures 6 and 7, we have shown the average value of the modulus of angular velocity of a vortex line around the instantaneous centroid of co-ordinates of the vortices, and its maximum and minimum values at different times. Figure 6 corresponds to $N = 50$, $X = 4$, and the vortices are bounded in a coaxial cylinder.

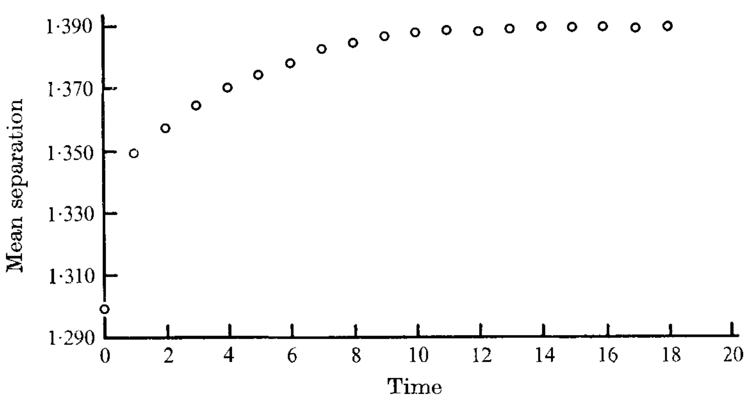

FIGURE 4. The mean separation of the vortices is plotted against time, when $N = 50$, $X = 4$ and $\Delta\tau = 0{\cdot}01$. The initial perturbation is of type A.

Figure 7 corresponds to $N = 50$, $X = \infty$. These two figures show that the motion of the vortices is approximately a solid body rotation.

When the number of vortices was large and the boundary was very close to the vortex polygon, we had to choose a very fine time step (usually of order 10^{-4} or even less) to get an acceptable solution, and the required computer time became too large. Also, when the polygon is very close to the boundary, the nonlinear behaviour is very complicated. The reason is that when a vortex is near

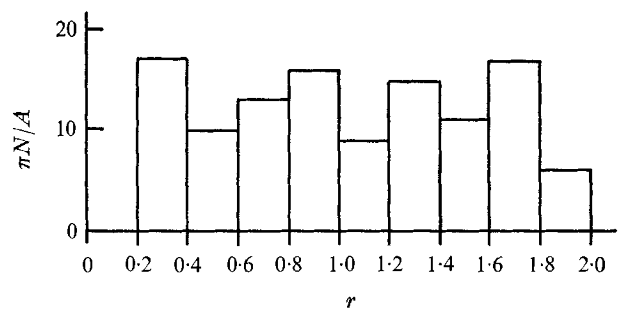

FIGURE 5. Distribution of vortices *versus* radial distance, (r), when $N = 50$, $X = \infty$ and $\Delta\tau = 0{\cdot}04$ at $\tau = 20$.

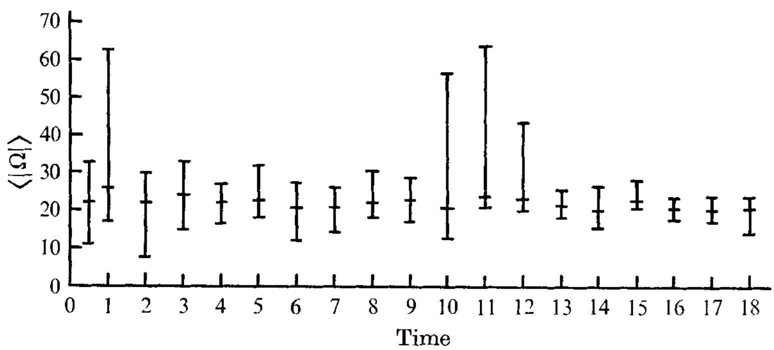

FIGURE 6. Average value of the modulus of angular velocity of vortices *versus* time is shown, when $N = 50$, $X = 4$ and $\Delta\tau = 0{\cdot}01$. Compare with figure 7.

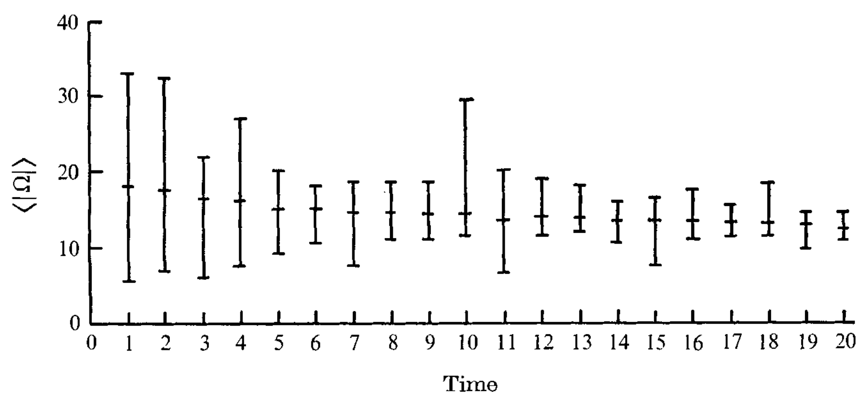

FIGURE 7. Average value of the modulus of angular velocity of vortices *versus* time is shown, when $N = 50$, $X = \infty$ and $\Delta\tau = 0{\cdot}04$.

the wall of the container, one term of the second summation in the non-linear differential equation (5) tends to become large, and therefore the time step, $\Delta\tau$, required for the numerical solution becomes too small and thereby reduces the practicability of the computation.

4. Conclusions

From the above numerical experiments, it seems plausible to conclude that a system of parallel line vortices will, due to non-linear interaction, become rearranged in such a way as to produce a more or less uniform distribution of vortices inside the fluid with an approximately constant mean separation. The average angular velocity of the rotation of the vortex lines about the instantaneous centroid of the vortex system remains approximately constant. This numerical result conforms with the conjecture of Raja Gopal (1964). Though we initially started with a regular arrangement of line vortices one can expect that the above conclusions will also hold for an initial irregular configuration of parallel line vortices because the initial state of such a system corresponds to an intermediate state of the evolution of the regular configuration that we studied.

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